NONSTATIONARY REGIME OF HEAT EXCHANGERS WITH INTERNAL HEAT SOURCES

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Analytic relations are obtained for determining the stability of thermoelectric devices and heat exchangers with heat sources whose strength is a linear function of temperature.

The stationary temperature distribution of a heat exchanger whose walls contain semiconductor thermocouples was investigated in [1]. It is required to determine whether the assumed stationary regime can always be established. In fact, as will be shown below, the presence of a heat source of variable strength may lead to instability of the system, i.e., to a monotonic rise in temperature.

In analyzing the stability of the system we make the same basic assumptions as in [1]. Moreover, it is assumed that the wall of the heat exchanger is sufficiently thin and its heat capacity negligible as compared with that of the heat transfer fluids.

We consider the case of countercurrent flow. Taking into account the variation of the enthalpy of the fluids in time, we can write the system of equations for the temperatures of the fluids $\tau_{1,2}$ and the wall surfaces $\tau'_{1,2}$ in the form

$$\frac{\partial \tau_1}{\partial \vartheta} + \frac{\partial \tau_1}{\partial \xi} = N_1(\tau_1' - \tau_1),$$
$$\gamma \frac{\partial \tau_2}{\partial \vartheta} - \frac{\partial \tau_2}{\partial \xi} = N_2(\tau_2' - \tau_2); \qquad (1)$$

$$\mathrm{Bi}_1(\tau_1 - \tau_1') = \nu\tau_1' - 0{,}5\nu^2 - (\tau_2' - \tau_1'),$$
$$\mathrm{Bi}_2(\tau_2' - \tau_2) = \nu\tau_2' + 0{,}5\nu^2 - (\tau_2' - \tau_1'). \qquad (2)$$

Here, it is assumed that $\gamma \le 1$ (i.e., $v_2 \ge v_1$). Otherwise it is necessary to introduce the inverse velocity ratio $\gamma' = v_2/v_1$ into the first equation of system (1); this substitution does not affect the result of the stability analysis. The relationship between v_1 and v_2 must be taken into account in order to calculate the dimensionless time with respect to the maximum flow velocity, which makes it possible to use an operational method of solution.

We assume that at the initial instant the temperature of both fluids is constant along the length of the heat exchanger and equal to the temperature at the point of admission, where it is kept constant in time:

$$\tau_1(\xi)|_{\vartheta=0} = \tau_1(\theta)|_{\xi=0} = \tau_{10},$$
$$\tau_2(\xi)|_{\vartheta=0} = \tau_2(\theta)|_{\xi=0} = \tau_{20}. \qquad (3)$$

To solve the problem of the stability of the temperature regime we use the Laplace transform method, after first eliminating the variables τ'_1 and τ'_2 from the starting equations. With (3) taken into account, the transformed functions $\bar{\tau}_{1,2}(\xi, p)$ should satisfy Eqs. (4) and the boundary conditions (5):

$$\frac{\partial \bar{\tau}_1}{\partial \xi} + \left\{p + \frac{N_1}{F}[\mathrm{Bi}_2(1+\nu) - \nu^2]\right\}\bar{\tau}_1 - \frac{N_1 \mathrm{Bi}_2}{F}\bar{\tau}_2 = \tau_{10} + \frac{N_1\nu^2}{2pF}(\mathrm{Bi}_2 - \nu + 2),$$
$$-\frac{\partial \bar{\tau}_2}{\partial \xi} + \left\{\gamma p + \frac{N_2}{F}[\mathrm{Bi}_1(1-\nu) - \nu^2]\right\}\bar{\tau}_2 - \frac{N_2 \mathrm{Bi}_1}{F}\bar{\tau}_1 = \gamma\tau_{20} + \frac{N_2\nu^2}{2pF}(\mathrm{Bi}_2 + \nu + 2); \qquad (4)$$

$$\bar{\tau}_1|_{\xi=0} = \frac{\tau_{10}}{p}, \quad \bar{\tau}_2|_{\xi=1} = \frac{\tau_{20}}{p}, \tag{5}$$

where

$$F = \mathrm{Bi}_1\mathrm{Bi}_2 + \mathrm{Bi}_1 + \mathrm{Bi}_2 + \nu(\mathrm{Bi}_2 - \mathrm{Bi}_1) - \nu^2.$$

The solution of system (4) for $\bar{\tau}_1(\xi, p)$ can be written in the form

$$\bar{\tau}_1(\xi, p) = \{A(p)\,\mathrm{sh}\sqrt{Y^2(p) - \alpha^2}\xi + B(p)\,\mathrm{sh}\sqrt{Y^2(p) - \alpha^2}(1-\xi)\}\exp[X(p)\xi] + R(p). \tag{6}$$

Here, we have introduced the notation

$$X(p) = -\frac{1}{2}\left\{p(1-\gamma) + \frac{1}{F}[N_1\mathrm{Bi}_2 - N_2\mathrm{Bi}_1 + \nu(N_1\mathrm{Bi}_2 + N_2\mathrm{Bi}_1) + \nu^2(N_2 - N_1)]\right\};$$

$$Y(p) = \frac{1}{2}\left\{p(1+\gamma) + \frac{1}{F}[N_1\mathrm{Bi}_2 + N_2\mathrm{Bi}_1 + \nu(N_1\mathrm{Bi}_2 - N_2\mathrm{Bi}_1) - \nu^2(N_1 + N_2)]\right\};$$

$$R(p) = \frac{L(p)}{M(p)};$$

$$M(p) = pF(p^2\gamma F + p\{N_2[\mathrm{Bi}_1(1-\nu) - \nu^2] + \gamma N_1[\mathrm{Bi}_2(1+\nu) - \nu^2]\} - \nu^2 N_1N_2);$$

$$A(p) = N(p)\{M(p)\,\mathrm{sh}\sqrt{Y^2(p) - \alpha^2}\,[Y(p)\,\mathrm{sh}\sqrt{Y^2(p) - \alpha^2} + \sqrt{Y^2(p) - \alpha^2}\,\mathrm{ch}\sqrt{Y^2(p) - \alpha^2}]\}^{-1};$$

$$B(p) = \frac{P(p)}{M(p)\,\mathrm{sh}\sqrt{Y^2(p) - \alpha^2}};$$

$$\alpha^2 = \frac{N_1N_2\,\mathrm{Bi}_1\mathrm{Bi}_2}{F^2}.$$

The functions L(p), N(p), P(p) are entire and transcendental (the expressions have been omitted because of their clumsiness).

The right side of Eq. (6) is a unique function of p satisfying the conditions of Jordan's lemma. This makes it possible, in going over to the time region, to use a closed integration path in the plane p and employ the residue theorem. The multiplier in the form $\exp[X(p)\xi]$ determines the variation of $\tau_1(\xi, \vartheta)$ along the ξ-coordinate and also the time lag, which depends on the ξ-coordinate, and does not affect the stability. The poles of expression (6) are the point $p = 0$, which determines the stationary distribution $\tau_1(\xi)$, and the roots of the equation

$$Y(p)\,\mathrm{sh}\sqrt{Y^2(p) - \alpha^2} + \sqrt{Y^2(p) - \alpha^2}\,\mathrm{ch}\sqrt{Y^2(p) - \alpha^2} = 0. \tag{7}$$

Thus, the starting system is stable if the roots of Eq. (7) have a negative real part. An equation of type (7) is fundamental to the solution of the stability problem not only for thermoelectric devices but also for heat exchangers with internal energy sources of another nature, for example, in the case of heat release due to chemical reactions in the fluid. In [2] an attempt was made to examine the stability of a heat exchanger with a chemical heat source proportional to the temperature. However, the author only hints at the possibility of an unstable state without giving the corresponding criteria.

Obviously, Eq. (7) has a root at the points $Y^2(p) = \alpha^2$, which correspond to the two values of p:

$$p_{1,2} = [-(N_1\mathrm{Bi}_2 + N_2\mathrm{Bi}_1) - \nu(N_1\mathrm{Bi}_2 - N_2\mathrm{Bi}_1) + \nu^2(N_1 + N_2) \pm 2\sqrt{N_1N_2\mathrm{Bi}_1\mathrm{Bi}_2}]\left[(1+\gamma)F\right]^{-1}. \tag{8}$$

The condition $F > 0$, which limits the current density, is always satisfied for actual thermoelectric cooling devices. Then from (8) there follows the necessary condition of stability

$$\nu^2 (N_1 + N_2) - \nu (N_1 \mathrm{Bi}_2 - N_2 \mathrm{Bi}_1) - (\sqrt{N_1 \mathrm{Bi}_2} - \sqrt{N_2 \mathrm{Bi}_1})^2 < 0. \tag{9}$$

We show that the other roots of Eq. (7) lie in the plane p to the left of the root p_1 and, hence, that inequality (9) is also a sufficient condition of stability.

We introduce the notation $(Y^2(p) - \alpha^2)^{1/2} = a + bi$, $Y(p) = c + di$, where a, b, c, d are arbitrary real numbers. It is easy to show that for the roots of Eq. (7)

$$\exp(4a) = \frac{a^2 + b^2 + c^2 + d^2 - 2ac - 2bd}{a^2 + b^2 + c^2 + d^2 + 2ac + 2bd}. \tag{10}$$

Moreover, by definition

$$c^2 - d^2 - \alpha^2 = a^2 - b^2, \tag{11}$$

$$cd = ab. \tag{12}$$

For nonzero a, b, c and d from relations (12) and (10, respectively, there follow the sign rules sign (ac) = = sign(ac + bd) and − sign a = sign(ac + bd), whence $c < 0$. At $\mathrm{Re}Y(p) < 0$ the corresponding roots of Eq. (7) lie in the complex plane to the left of the root p_1. When some of the quantities a, b, c and d vanish, there are two possible cases of satisfaction of Eqs. (10), (11) and (12): 1) $a = 0$, $b = 0$, $d = 0$; 2) $a = 0$, $b \neq 0$, $d = 0$. The first of these gives for Eq. (7) the known pair of roots $p_{1,2}$, and the second roots lying to the left of the root p_1. This proves the sufficiency of condition (9).

Inequality (9) is satisfied on the interval $0 < \nu < \nu_0$. It is convenient to write the maximum current density ν_0 permissible for a stable system in terms of the water equivalents $W_{1,2}$ of the fluids and the parameters $\beta_1 = (\mathrm{Bi}_1)^{-1}$, $\beta_2 = (\mathrm{Bi}_2)^{-1}$:

$$\nu_0 = \frac{1}{2} [\beta_1 W_1 + \beta_2 W_2]^{-1} \{W_2 - W_1 + [(W_2 - W_1)^2 + 4(\beta_1 W_1 + \beta_2 W_2)(\sqrt{W_2} - \sqrt{W_1})^2]^{\frac{1}{2}}\}. \tag{13}$$

The value ν_0 coincides with the current defining the limit of occurence of a periodic stationary temperature distribution along the length ξ (Eq. (13) of [1]). Actually, owing to the instability of the system such a stationary temperature distribution cannot develop.

When $W_1 = W_2$ $\nu_0 = 0$; therefore, in the presence of a symmetrical flow over the wall of the heat exchanger the temperature regime is always unstable. An example of an unstable system with symmetrical flows is the regenerative heat exchanger considered in [3], which, as follows from the above reasoning, cannot operate as a cooling device.

The region of stability for a parallel flow can be similarly analyzed. It is found that the parallel-flow heat exchanger is a stable system at any values of the parameters.

The transient regime in counterflow heat exchangers is characterized by temperature oscillations; in the unstable regime the amplitude of the oscillations increases with time. Under actual conditions, an indefinite increase in temperature is, of course, prevented by the finite power of the current source and heat losses to the surrounding medium.

NOTATION

$\tau = zT$ is the dimensionless temperature; T is the temperature in the fluid flow; T' is the temperature of the junctions; T_0 is the inlet temperature of the heat-transfer agent: $z = \alpha^2/\rho\lambda$; α, λ, ρ are the reduced thermal emf, thermal conductivity, and resistivity; S, d, and l are the area, thickness and length of module; $\xi = x/l$ is the dimensionless coordinate; x is the coordinate along the flow; $\nu = (\alpha d/\lambda) j$ is the dimensionless current; j is the current density; $\mathrm{Bi}_{1,2} = 1/\beta_{1,2} = a_{1,2}\, d/\lambda$ is the Biot number; a is the heat-transfer coefficient with allowance for the radiator per unit area of the module; $N_{1,2} = Sa_{1,2}/W_{1,2}$ is the dimensionless area; W is the water equivalent of the flow; $\vartheta = tv_1/d$ is the dimensionless time; t is the time; $v_{1,2}$ are the flow velocities; $\gamma = 1/\gamma' = v_1/v_2$. The subscripts 1 and 2 relate to the cooled and heated heat-transfer agents, respectively.

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